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DIFFERENTIATION FORMULAS FOR STOCHASTIC INTEGRALS IN THE PLANE*

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For a one-parameter process of the form $X_t = X_0 + \int_0^t \phi_s \, dW_s + \int_0^t \psi_s \, ds$, where W is a Wiener process and $\int \phi \, dW$ is a stochastic integral, a twice continuously differentiable function $f(X_t)$ is again expressible as the sum of a stochastic integral and an ordinary integral via the Ito differentiation formula. In this paper we present a generalization for the stochastic integrals associated with a two-parameter Wiener process.

Let $\{W_z, z \in \mathbf{R}_+^2\}$ be a Wiener process with a two-dimensional parameter. Erstwhile, we have defined stochastic integrals $\int \phi \, dW$ and $\int \psi \, dW \, dW$, as well as mixed integrals $\int h \, dz \, dW$ and $\int g \, dW \, dz$. Now, let X_z be a two-parameter process defined by the sum of these four integrals and an ordinary Lebesgue integral. The objective of this paper is to represent a suitably differentiable function $f(X_z)$ as such a sum once again. In the process we will also derive the (basically one-dimensional) differentiation formulas of $f(X_z)$ on increasing paths in $\mathbf{R}_+^2$.

differentiation formulas	Ito lemma
martingales in the plane	stochastic integrals
two-parameter Wiener process	

1. Introduction

Let $\mathbf{R}_+^2$ denote the positive quadrant of the plane and let $\{W_z, z \in \mathbf{R}_+^2\}$ be a two-parameter Wiener process. Stochastic integrals of the form

$$I_1 = \int \phi_z \, dW_z$$

were defined in [1,3,9], and stochastic integrals of a second type

$$I_2 = \int \psi_{z,z'} \, dW_z \, dW_{z'}$$

were introduced in [4] where it was shown that every square-integrable two-parameter martingale generated by a Wiener process could be expressed as a sum $I_1 + I_2$. In deriving this result, we obtained a differential formula for those transformations $f(W_z, z)$ which are themselves martingales. While this formula has already found some applications [5], it is inadequate for a general calculus.

The natural question is the following: If we define a process X_z as the sum of a

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Lebesgue integral plus stochastic integrals of the first and second types, and if $f(x, z)$ is a suitably differentiable function, can $f(X_z, z)$ be expressed as such a sum of three integrals once again? The answer, interestingly, is no. For a complete generalization of the Ito lemma, we need the mixed integrals introduced in [6]. The purpose of this paper is to derive the general differentiation formula and some related results.

2. Notations and preliminaries

Let $a = (a_1, a_2)$ and $b = (b_1, b_2)$ be two points in the positive quadrant $\mathbf{R}_+^2$. We denote $a < b$ if $a_1 \leq b_1$ and $a_2 \leq b_2$, $a << b$ if $a_1 < b_1$ and $a_2 < b_2$, $a \barwedge b$ if $a_1 \leq b_1$ and $a_2 \geq b_2$, $a \doublebarwedge b$ if $a_1 < b_1$ and $a_2 > b_2$. Furthermore, we shall adopt the notations:

$$a \otimes b = (a_1, b_2),$$

$$a \wedge b = (\min(a_1, b_1), \min(a_2, b_2)),$$

$$a \vee b = (\max(a_1, b_1), \max(a_2, b_2)).$$

Note that if $a \barwedge b$ then $a \otimes b = a \wedge b$, if $b \barwedge a$ then $a \otimes b = a \vee b$, and that $a \otimes b \otimes c = a \otimes c$.

For a fixed point $a \in \mathbf{R}_+^2$, R_a will denote the rectangle $\{z : z \in \mathbf{R}_+^2, z < a\}$. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space, and let $\{\mathcal{F}_z, z \in R_a\}$ be a family of σ-subfields such that it is increasing, right-continuous (w.r.t. $>$) and

For each z, $\mathcal{F}_z^1 = \mathcal{F}_{z \otimes a}$ and $\mathcal{F}_z^2 = \mathcal{F}_{a \otimes z}$ are conditionally independent given $\mathcal{F}_z$.

A process $\{M_z, \mathcal{F}_z, z \in R_a\}$ is said to be a *martingale* if:

(1) for each z, M_z is $\mathcal{F}_z$-measurable,

(2) for each z $E|M_z| < \infty$,

(3) $z < z'$ implies $E(M_{z'} | \mathcal{F}_z)$ equals M_z almost surely.

In [2] Cairoli and Walsh introduced the concepts of strong and weak martingales, and 1 and 2 martingales. Adapted 1 and 2 martingales were introduced in [6]. It can be shown that (see [2] and [6]) with these definitions, a strong martingale is also a martingale, a process is a martingale if and only if it is both an adapted 1-martingale and an adapted 2-martingale, and adapted one and two martingales are also weak martingales.

3. Stochastic integrals

Let $\{W_z, \mathcal{F}_z, z \in R_a\}$ be a Wiener process. Let $\{\phi_z, z \in R_a\}$ be a process such that:

(a) ϕ is a bimeasurable function of (ω, z).

(b) $$\int_{R_a} \mathbf{E}\,\phi_\zeta^2 \mathrm{d}\zeta < \infty \tag{3.1}$$

and for each z

either (c_0) ϕ_z is $\mathcal{F}_z$-measurable,

or (c_1) ϕ_z is $\mathcal{F}_z^1$-measurable,

or (c_2) ϕ_z is $\mathcal{F}_z^2$-measurable.

Let $\mathcal{H}_i$ denote the space of ϕ satsifying (a), (b) and (c_i). For $\phi \in \mathcal{H}_i$, $i = 0, 1, 2$, the stochatic integral $\int_{R_a} \phi_\zeta \, dW_\zeta$ is well-defined. If we define

$$(\phi \circ W)_z = \int_{R_z} \phi_\zeta \, dW_\zeta = \int_{R_a} I(\zeta < z)\phi_\zeta \, dW_\zeta \tag{3.2}$$

(I = indicator function) then the process $\phi \circ W$ is a strong martingale if $\phi \in \mathcal{H}_0$, a 1-martingale if $\phi \in \mathcal{H}_1$ and a 2-martingale if $\phi \in \mathcal{H}_2$. Furthermore, define

$$X_z = (\phi \circ W)_z(\psi \circ W)_z - \int_{R_z} \phi_\zeta \psi_\zeta \, d\zeta. \tag{3.3}$$

Then X is a martingale if $\phi, \psi \in \mathcal{H}_0$, a 1-martingale if $\phi, \psi \in \mathcal{H}_1$, and a 2-martingale if $\phi, \psi \in \mathcal{H}_2$. In all cases continuous versions can be chosen [2].

Proposition 3.1. *Let $\{X_z, z \in R_a\}$ be a process defined by*

$$X_z = X_0 + \int_{R_z} f(z, \zeta) \, dW_\zeta$$

where X_0 is $\mathcal{F}_0$-measurable and f satisfies the conditions

$$\begin{aligned}
&\text{(a)} && f(z,\zeta) = 0 \text{ unless } \zeta < z,\\
&\text{(b)} && f(z,\zeta) = f(\zeta \otimes z, \zeta),\\
&\text{((b')} && f(z,\zeta) = f(z \otimes \zeta, \zeta)),\\
&\text{(c)} && \textit{for each } z \in R_a,\ f(z,\cdot) \in \mathcal{H}_1,\\
&\text{((c')} && f(z,\cdot) \in \mathcal{H}_2).
\end{aligned} \tag{3.4}$$

Then, X_z is an adapted 1-martingale (respectively, 2-martingale).

Remark. Except for notational differences and an explicit display of the dependence of the integrand on the domain of integration, Proposition 3.1 is a restatement of Proposition 2.3 of Cairoli and Walsh [2].

Next, let denote the space of functions $\psi(\omega, \zeta, \zeta')$ which satisfy:

$$\begin{aligned}
&\text{(a)} && \psi \text{ is a measurable process and for each } (\zeta, \zeta')\ \psi_{\zeta,\zeta'} \text{ is } \mathcal{F}_{\zeta \vee \zeta'}\text{-measurable.}\\
&\text{(b)} && \int_{R_a \times R_a} I(\zeta \mathrel{\bar{\wedge}} \zeta') \mathrm{E}\psi^2_{\zeta,\zeta'} \, d\zeta \, d\zeta' < \infty.
\end{aligned} \tag{3.5}$$

For such functions the stochastic integrals

$$
\begin{aligned}
X_z &= \int_{R_z \times R_z} \psi_{\zeta,\zeta'} \mathrm{d}W_\zeta \mathrm{d}W_{\zeta'} \\
Y_{1z} &= \int_{R_z \times R_z} \psi_{\zeta,\zeta'} \mathrm{d}\zeta \mathrm{d}W_{\zeta'} \\
Y_{2z} &= \int_{R_z \times R_z} \psi_{\zeta,\zeta'} \mathrm{d}W_\zeta \mathrm{d}\zeta'
\end{aligned}
\tag{3.6}
$$

are well defined for all z in R_a and X, Y_1, Y_2 are respectively a martingale, and an adapted i-martingale ($i = 1,2$) for which sample-continuous versions can be chosen [6]. We note that these integrals are defined in such a way that only the values of the integrand on $z \barwedge z'$ have an effect on the integral.

Proposition 3.2. *Let $\psi \in \mathscr{H}$ and define X, Y_1, and Y_2 by (3.6). Furthermore, let*

$$
\begin{aligned}
f_1(z,\zeta') &= \int_{R_z} I(\zeta \barwedge \zeta')\psi_{\zeta,\zeta'} \mathrm{d}W_\zeta, \\
f_2(z,\zeta) &= \int_{R_z} I(\zeta \barwedge \zeta')\psi_{\zeta,\zeta'} \mathrm{d}W_{\zeta'}, \\
g_1(z,\zeta') &= \int_{R_z} I(\zeta \barwedge \zeta')\psi_{\zeta,\zeta'} \mathrm{d}\zeta, \\
g_2(z,\zeta) &= \int_{R_z} I(\zeta \barwedge \zeta')\psi_{\zeta,\zeta'} \mathrm{d}\zeta'.
\end{aligned}
\tag{3.7}
$$

Then,

$$X_z = \int_{R_z} f_1(z,\zeta') \mathrm{d}W_{\zeta'} = \int_{R_z} f_2(z,\zeta) \mathrm{d}W_\zeta \tag{3.8}$$

$$Y_{1z} = \int_{R_z} g_1(z,\zeta') \mathrm{d}W_{\zeta'} = \int_{R_z} f_2(z,\zeta) \mathrm{d}\zeta \tag{3.9}$$

$$Y_{2z} = \int_{R_z} g_2(z,\zeta) \mathrm{d}W_\zeta = \int_{R_z} f_1(z,\zeta') \mathrm{d}\zeta'. \tag{3.10}$$

Remark. Proposition 3.2 might be viewed as a stochastic Fubini's theorem, and (3.8) is Theorem 2.6 of [2]. (3.9) and (3.10) can be proved in a similar way as (3.8).

If condition (b) of (3.1) is replaced by

$$\int_{R_a} \phi_z^2 \mathrm{d}z < \infty \quad \text{almost surely}$$

the stochastic integral (3.2) can still be defined. Sample-continuous version can again be chosen, but like the one-parameter case the martingale properties must now be replaced by appropriate local martingale properties. Similarly, the stochastic integrals of (3.6) remain well defined if condition (b) of (3.5) is replaced by a condition of almost-sure-integrability. Sample-continuity is again assured. These recent results are proved in [7, 8].

4. Formulas on partial differentiation

In [6] we have shown that under suitable differentiability conditions, every weak martingale can be represented as the sum of stochastic integrals of the four types. If we call processes of the form $X_z = (\text{weak martingale}) + \int_{R_z} u_\zeta \, d\zeta$ *weak semi-martingales*, then our principal result (Section 5) will be a representation of sufficiently smooth functions $F(X_z)$ as weak semi-martingales once again, via a differentiation formula.

Suppose that $\{X_z, z \in R_a\}$ is a process of the form

$$X_z = X_0 + \int_{R_z} f(z,\zeta)\,dW_\zeta + \int_{R_z} u(z,\zeta)\,d\zeta \tag{4.1}$$

where f satisfies the conditions of Proposition 3.1 to make the stochastic integral $\int_{R_z} f(z,\zeta)dW_\zeta$ an adapted 1-martingale and u satisfies $u(z,\zeta) = u(\zeta \otimes z, \zeta)$. Let $z = (s,t)$ and $\zeta = (\sigma,\tau)$. Then $\zeta \otimes z = (\sigma, t)$ and by setting $f((\sigma,t),(\sigma,\tau)) = \tilde{f}(t;\sigma,\tau)$ and $u((\sigma,t),(\sigma,\tau)) = u(t;\sigma,\tau)$, we can reexpress X_z as

$$X_{s,t} = X_0 + \int_{R_{s,t}} \tilde{f}(t,\zeta)\,dW_\zeta + \int_{R_{s,t}} \tilde{u}(t,\zeta)\,d\zeta. \tag{4.2}$$

$X_{s,t}$ is a one-parameter semimartingale in s for each t. Rewriting it as

$$X_{s,t} = X_0 + M_s^t + \int_0^s \left[\int_0^t \tilde{u}(t,\sigma,\tau)\,d\tau\right] d\sigma \tag{4.3}$$

we get the one-parameter formula

$$\begin{aligned} F(X_{s,t}) = F(X_0) &+ \int_0^s F'(X_{\sigma,t})\left\{ dM_\sigma^t + \left[\int_0^t \tilde{u}(t,\sigma,\tau)\,d\tau\right] d\sigma \right\} \\ &+ \tfrac{1}{2}\int_0^s F''(X_{\sigma,t})\,d\langle M^t, M^t\rangle_\sigma \end{aligned} \tag{4.4}$$

for any twice continuously differentiable F. Equation (4.4) can be rewritten as

$$\begin{aligned} F(X_{s,t}) = F(X_0) &+ \int_0^s \int_0^t F'(X_{\sigma,t})\{\tilde{f}(t;\sigma,\tau)\,dW_{\sigma,\tau} + \tilde{u}(t;\sigma,\tau)\,d\sigma\,d\tau\} \\ &+ \tfrac{1}{2}\int_0^s \int_0^t F''(X_{\sigma,t})\tilde{f}^2(t;\sigma,\tau)\,d\sigma\,d\tau \end{aligned}$$

or

$$\begin{aligned} F(X_z) = F(X_0) &+ \int_{R_z} F'(X_{\zeta\otimes z})\{f(\zeta\otimes z,\zeta)\,dW_\zeta + u(\zeta\otimes z,\zeta)\,d\zeta\} \\ &+ \tfrac{1}{2}\int_{R_z} F''(X_{\zeta\otimes z}) f^2(\zeta\otimes z,\zeta)\,d\zeta \end{aligned}$$

$$= F(X_0) + \int_{R_z} F'(X_{\zeta\otimes z})\{f(z,\zeta)\mathrm{d}W_\zeta + \ 'z,\zeta)\mathrm{d}\zeta\} \tag{4.5}$$

$$+\tfrac{1}{2}\int_{R_z} F''(X_{\zeta\otimes z})f^2(z,\zeta)\mathrm{d}\zeta.$$

To summarize, we have the following result:

Proposition 4.1. *Let X_{kz}, $z \in R_a$, $k = 1,2,\ldots,n$, be processes defined by*

$$X_{kz} = X_{k0} + \int_{R_z} f_k(z,\zeta)\mathrm{d}W_\zeta + \int_{R_z} u_k(z,\zeta)\mathrm{d}\zeta. \tag{4.6}$$

Suppose that for each k, f satisfies the conditions of Proposition 3.1 to make the stochastic integral a 1-martingale and $u_k(z,\zeta) = u_k(\zeta\otimes z,\zeta)$. Let $X = (X_1, X_2, \ldots, X_n)$ and $F(X)$ be a function with continuous partials up to the second order. Then,

$$\begin{aligned} F(X_z) = F(X_0) &+ \sum_k \int_{R_z} F_k(X_{\zeta\otimes z})[f_k(z,\zeta)\mathrm{d}W_\zeta + u_k(z,\zeta)\mathrm{d}\zeta] \\ &+ \tfrac{1}{2}\sum_{k,l}\int_{R_z} F_{kl}(X_{\zeta\otimes z})f_k(z,\zeta)f_l(z,\zeta)\mathrm{d}\zeta \end{aligned} \tag{4.7}$$

where F_k and Fk_l denote partial derivatives. Alternatively, if f_k satisfy the conditions of Proposition 3.1 to make the stochastic integral a 2-martingale and $u_k(z,\zeta) = u_k(z\otimes\zeta,\zeta)$, then

$$\begin{aligned} F(X_z) = F(X_0) &+ \sum_k \int_{R_z} F_k(X_{z\otimes\zeta})[f_k(z,\zeta)\mathrm{d}W_\zeta + u_k(z,\zeta)\mathrm{d}\zeta] \\ &+ \tfrac{1}{2}\sum_{k,l}\int_{R_z} F_{kl}(X_{z\otimes\zeta})f_k(z,\zeta)f_l(z,\zeta)\mathrm{d}\zeta. \end{aligned} \tag{4.7'}$$

An important special case of a process X which is of the form (4.6) is given by

$$\begin{aligned} X_z = \int_{R_z} \theta_\zeta\,\mathrm{d}\zeta &+ \int_{R_z} \phi_\zeta\,\mathrm{d}W_\zeta + \int_{R_z\times R_z} \psi_{\zeta,\zeta'}\mathrm{d}W_\zeta\mathrm{d}W_{\zeta'} \\ &+ \int_{R_z\times R_z} g_{\zeta,\zeta'}\mathrm{d}\zeta\mathrm{d}W_{\zeta'} + \int_{R_z\times R_z} h_{\zeta,\zeta'}\mathrm{d}W_\zeta\mathrm{d}\zeta' \end{aligned} \tag{4.8}$$

which can be written in the form of (4.6) in two ways, with either

$$\begin{aligned} f(z,\zeta) &= \phi_\zeta + \int_{R_z} I(\zeta' \barwedge \zeta)[\psi_{\zeta',\zeta}\mathrm{d}W_{\zeta'} + g_{\zeta',\zeta}\mathrm{d}\zeta'] \\ u(z,\zeta) &= \theta_\zeta + \int_{R_z} I(\zeta' \barwedge \zeta)h_{\zeta',\zeta}\mathrm{d}W_{\zeta'} \end{aligned} \tag{4.9}$$

or

$$\begin{aligned} f(z,\zeta) &= \phi_\zeta + \int_{R_z} I(\zeta \barwedge \zeta')[\psi_{\zeta,\zeta'}\mathrm{d}W_{\zeta'} + h_{\zeta,\zeta'}\mathrm{d}\zeta'] \\ u(z,\zeta) &= \theta_\zeta + \int_{R_z} I(\zeta \barwedge \zeta')g_{\zeta,\zeta'}\mathrm{d}W_{\zeta'}. \end{aligned} \tag{4.10}$$

It is easy to verify that in the first case because of the term $I(\zeta' \curlywedge \zeta)$, $f(z,\zeta) = f(\zeta \otimes z, \zeta)$ and $u(z,\zeta) = u(\zeta \otimes z, \zeta)$ and for the second case $f(z,\zeta) = f(z \otimes \zeta, \zeta)$ and $u(z,\zeta) = u(z \otimes \zeta, \zeta)$.

We note that for a fixed ζ, $f(z,\zeta)$ and $u(z,\zeta)$ as given by (4.9) and (4.10) are adapted 1 and 2 semi-martingales, and differentiation rules apply once again.

5. The Ito lemma for stochastic integrals in the plane

Let Z_{kz}, $z \in R_a$, $k = 1,2,\ldots,m$, be processes defined by

$$\begin{aligned} X_{kz} = Z_{k0} &+ \int_{R_z} \theta_{k\zeta}\,\mathrm{d}\zeta + \int_{R_z} \phi_{k\zeta}\,\mathrm{d}W_\zeta + \int_{R_z \times R_z} \psi_{k,\zeta,\zeta'}\,\mathrm{d}W_\zeta\,\mathrm{d}W_{\zeta'} \\ &+ \int_{R_z \times R_z} f_{k,\zeta,\zeta'}\,\mathrm{d}\zeta\,\mathrm{d}W_{\zeta'} + \int_{R_z \times R_z} g_{k,\zeta,\zeta'}\,\mathrm{d}W_\zeta\,\mathrm{d}\zeta'. \end{aligned} \tag{5.1}$$

If we set

$$u_k(z,\zeta') = \phi_{k\zeta'} + \int_{R_z} I(\zeta \curlywedge \zeta')\psi_{k,\zeta,\zeta'}\,\mathrm{d}W_\zeta + \int_{R_z} I(\zeta \curlywedge \zeta') f_{k,\zeta,\zeta'}\,\mathrm{d}\zeta \tag{5.2}$$

and

$$v_k(z,\zeta') = \theta_{k\zeta'} + \int_{R_z} I(\zeta \curlywedge \zeta') g_{k,\zeta,\zeta'}\,\mathrm{d}W_\zeta, \tag{5.3}$$

then (5.1) can be rewritten as

$$X_{kz} = X_{k0} + \int_{R_z} u_k(z,\zeta')\,\mathrm{d}W_{\zeta'} + \int_{R_z} v_k(z,\zeta')\,\mathrm{d}\zeta' \tag{5.4}$$

which is of the same form as (4.6), and u_k and v_k satisfy the conditions for (4.7). Therefore, we have

$$\begin{aligned} F(X_z) = F(Z_0) &+ \sum_k \int_{R_z} F_k(X_{\zeta'\otimes z})[u_k(z,\zeta')\,\mathrm{d}W_{\zeta'} + v_k(z,\zeta')\,\mathrm{d}\zeta'] \\ &+ \tfrac{1}{2}\sum_{k,l} \int_{R_z} F_{kl}(X_{\zeta'\otimes z}) u_k(z,\zeta') u_l(z,\zeta')\,\mathrm{d}\zeta'. \end{aligned} \tag{5.5}$$

Now, (5.1) can also be reexpressed as

$$X_{kz} = X_{k0} + \int_{R_z} [\tilde{u}_k(z,\zeta)\,\mathrm{d}W_\zeta + \tilde{v}_k(z,\zeta)\,\mathrm{d}\zeta] \tag{5.6}$$

with $\tilde{u}_k$ and $\tilde{v}_k$ given by

$$\tilde{u}_k(z,\zeta) = \phi_{k\zeta} + \int_{R_z} I(\zeta \curlywedge \zeta')[\psi_{k,\zeta,\zeta'}\,\mathrm{d}W_{\zeta'} + g_{k,\zeta,\zeta'}\,\mathrm{d}\zeta'], \tag{5.7}$$

$$\tilde{v}_k(z,\zeta) = \theta_{k\zeta} + \int_{R_z} I(\zeta \curlywedge \zeta') f_{k,\zeta,\zeta'}\,\mathrm{d}W_{\zeta'}. \tag{5.8}$$

Observe that because of the term $I(\zeta \wedge \zeta')$ in the integrals $\tilde{u}_k(z,\zeta) = \tilde{u}_k(z \otimes \zeta, \zeta)$ and $\tilde{v}_k(z,\zeta) = v_k(z \otimes \zeta, \zeta)$. Therefore, for any fixed point ζ',

$$
\begin{aligned}
X_{k\zeta'\otimes z} - X_{k\zeta'} &= \int_{R_{\zeta'\otimes z} - R_{\zeta'}} [\tilde{u}_k(\zeta'\otimes\zeta,\zeta)\,\mathrm{d}W_\zeta + \tilde{v}_k(\zeta'\otimes\zeta,\zeta)\,\mathrm{d}\zeta] \\
&= \int_{R_z} I(\zeta \wedge \zeta')[\tilde{u}_k(\zeta'\otimes\zeta,\zeta)\,\mathrm{d}W_\zeta + v_k(\zeta'\otimes\zeta,\zeta)\,\mathrm{d}\zeta].
\end{aligned} \tag{5.9}
$$

The equations (5.2), (5.3) and (5.9) are all of the same form, viz.,

$$
Y(z,\zeta') = \alpha_{\zeta'} + \int_{R_z} I(\zeta \wedge \zeta')[\beta_{\zeta,\zeta'}\,\mathrm{d}W_\zeta + \gamma_{\zeta,\zeta'}\,\mathrm{d}\zeta] \tag{5.10}
$$

which is a 2-semimartingale for each fixed ζ'. Therefore, we can reexpress the integrands of (5.5) using (4.7), the differentiation formula for 2-semimartingales, e.g.,

$$
\begin{aligned}
F_k(X_{\zeta'\otimes z})u_k(z,\zeta') &= F_k(X_{\zeta'})\phi_{k\zeta'} \\
&\quad + \int_{R_z} I(\zeta \wedge \zeta')F_k(X_{\zeta'\otimes\zeta})[\psi_{k,\zeta,\zeta'}\,\mathrm{d}W_\zeta + f_{k,\zeta,\zeta'}\,\mathrm{d}\zeta] \\
&\quad + \int_{R_z} I(\zeta \wedge \zeta')u_k(\zeta'\otimes\zeta,\zeta') \sum_l F_{kl}(X_{\zeta'\otimes\zeta}) \\
&\qquad\qquad \times [\tilde{u}_l(\zeta'\otimes\zeta,\zeta)\,\mathrm{d}W_\zeta + \tilde{v}_k(\zeta'\otimes\zeta,\zeta)\,\mathrm{d}\zeta] \\
&\quad + \int_{R_z} I(\zeta \wedge \zeta')\left[\sum_l F_{kl}(X_{\zeta'\otimes\zeta})\psi_{k,\zeta,\zeta'}\tilde{u}_l(\zeta'\otimes\zeta,\zeta)\right]\mathrm{d}\zeta \\
&\quad + \tfrac{1}{2}\int_{R_z} I(\zeta \wedge \zeta')u_k(\zeta'\otimes\zeta,\zeta') \\
&\qquad\qquad \times \left[\sum_{l,m} F_{klm}(X_{\zeta'\otimes\zeta})\tilde{u}_l(\zeta'\otimes\zeta,\zeta)\tilde{u}_m(\zeta'\otimes\zeta,\zeta)\right]\mathrm{d}\zeta.
\end{aligned}
$$

If this tedious but straightforward procedure is applied to every term of the integrand in (5.5), we get the following:

Proposition 5.1. *Let X_{kz}, $z \in R_a$, $k = 1,2,\ldots,n$, be process defined by (5.1), where the integrands are almost surely bounded. Let $F(x)$, $x \in \mathbf{R}^n$, be a function with continuous mixed partials through the fourth order. Then,*

$$
\begin{aligned}
F(X_z) = F(X_0) &+ \int_{R_z} F_k(X_\zeta)[\phi_k\,\mathrm{d}W_\zeta + \theta_k\,\mathrm{d}\zeta] + \tfrac{1}{2}\int_{R_z} F_{kl}(X_\zeta)\phi_{k\zeta}\phi_{l\zeta}\,\mathrm{d}\zeta \\
&+ \int_{R_z\times R_z} [F_{kl}(X_{\zeta\vee\zeta'})u_k\tilde{u}_l + F_k(X_{\zeta\vee\zeta'})\psi_k]\,\mathrm{d}W_\zeta\,\mathrm{d}W_{\zeta'} \\
&+ \int_{R_z\times R_z} [F_k(X_{\zeta\vee\zeta'})f_k + F_{kl}(X_{\zeta\vee\zeta'})(u_k\tilde{v}_l + \psi_k\tilde{u}_l) \\
&\qquad + \tfrac{1}{2}F_{klm}(X_{\zeta\vee\zeta'})u_k\tilde{u}_l\tilde{u}_m]\,\mathrm{d}\zeta\,\mathrm{d}W_{\zeta'}
\end{aligned}
$$

$$+\int_{R_z\times R_z}[F_k(X_{\zeta\vee\zeta'})g_k+F_{kl}(X_{\zeta\vee\zeta'})(\tilde{u}_kv_l+\psi_ku_l)$$
$$+\tfrac{1}{2}F_{klm}(X_{\zeta\vee\zeta'})\tilde{u}_ku_lu_m]\,\mathrm{d}W_\zeta\,\mathrm{d}\zeta' \tag{5.11}$$
$$+\int_{R_z\times R_z}I(\zeta\;\bar{\wedge}\;\zeta')\{F_{kl}(X_{\zeta\vee\zeta'})(v_k\tilde{v}_l+g_k\tilde{u}_l+f_ku_l+\tfrac{1}{2}\psi_k\psi_l)$$
$$+F_{klm}(X_{\zeta\vee\zeta'})(u_k\tilde{u}_l\psi_m+\tfrac{1}{2}v_k\tilde{u}_l\tilde{u}_m+\tfrac{1}{2}\tilde{v}_ku_lu_m)$$
$$+\tfrac{1}{4}F_{klmp}(X_{\zeta\vee\zeta'})u_ku_l\tilde{u}_m\tilde{u}_p\}\,\mathrm{d}\zeta\,\mathrm{d}\zeta'$$

where u and v have arguments $(\zeta\vee\zeta',\zeta')$, $\tilde{u}$ and $\tilde{v}$ have arguments $(\zeta\vee\zeta',\zeta)$, ψ, f and g have arguments (ζ,ζ') and all repeated indices are summed from 1 to n.

Observe that we have made use of the relationship $\zeta\vee\zeta'=\zeta'\otimes\zeta$ if $\zeta\;\bar{\wedge}\;\zeta'$.

Because of its complexity, the final expression for the differentiation formula may not be as useful as the partial differentiation formulas which give rise to it. Specifically, we are referring to (5.5) and the equations (5.2), (5.3) and (5.9). Note that (5.5) is a representation of $F(X_z)$ as a 1-semimartingale, and (5.2), (5.3) and (5.9) provide a representation of the integrands as 2-semimartingales. An alternative form with the roles of 1 and 2 semimartingales reversed also exists. It is useful to summarize these results as follows.

$$\begin{aligned}F(X_z)&=F(X_0)+\int_{R_z}F_k(X_{\zeta'\otimes z})[u_k(z,\zeta')\,\mathrm{d}W_{\zeta'}+v_k(z,\zeta')\,\mathrm{d}\zeta']\\&\quad+\tfrac{1}{2}\int_{R_z}F_{kl}(X_{\zeta'\otimes z})u_k(z,\zeta')u_l(z,\zeta')\,\mathrm{d}\zeta'\\&=F(X_0)+\int_{R_z}F_k(X_{z\otimes\zeta})[\tilde{u}_k(z,\zeta)\,\mathrm{d}W_\zeta+\tilde{v}_k(z,\zeta)\,\mathrm{d}\zeta]\\&\quad+\tfrac{1}{2}\int_{R_z}F_{kl}(X_{z\otimes\zeta})\tilde{u}_k(z,\zeta)\tilde{u}_l(z,\zeta)\,\mathrm{d}\zeta\end{aligned}\tag{5.12}$$

$$\begin{aligned}X_{k\zeta'\otimes z}&=X_{k\zeta'}+\int_{R_z}I(\zeta\;\bar{\wedge}\;\zeta')[\tilde{u}_k(\zeta'\otimes\zeta,\zeta)\,\mathrm{d}W_\zeta+\tilde{v}_k(\zeta'\otimes\zeta,\zeta)\,\mathrm{d}\zeta]\\X_{kz\otimes\zeta}&=X_{k\zeta}+\int_{R_z}I(\zeta\;\bar{\wedge}\;\zeta')[u_k(\zeta'\otimes\zeta,\zeta')\,\mathrm{d}W_{\zeta'}+v_k(\zeta'\otimes\zeta,\zeta')\,\mathrm{d}\zeta']\end{aligned}\tag{5.13}$$

$$\begin{aligned}u_k(z,\zeta')&=\phi_{k\zeta'}+\int_{R_z}I(\zeta\;\bar{\wedge}\;\zeta')\psi_{k,\zeta,\zeta'}\,\mathrm{d}W_\zeta+\int_{R_z}I(\zeta\;\bar{\wedge}\;\zeta')f_{k,\zeta,\zeta'}\,\mathrm{d}\zeta\\\tilde{u}_k(z,\zeta)&=\phi_{k\zeta}+\int_{R_z}I(\zeta\;\bar{\wedge}\;\zeta')\psi_{k,\zeta,\zeta'}\,\mathrm{d}W_{\zeta'}+\int_{R_z}I(\zeta\;\bar{\wedge}\;\zeta')g_{k,\zeta,\zeta'}\,\mathrm{d}\zeta'\end{aligned}\tag{5.14}$$

$$\tilde{v}_k(z,\zeta') = \theta_{k\zeta'} + \int_{R_z} I(\zeta \curlywedge \zeta') g_{k,\zeta,\zeta'} \mathrm{d}W_\zeta$$

$$\tilde{v}_k(z,\zeta) = \theta_{k\zeta} + \int_{R_z} I(\zeta \curlywedge \zeta') f_{k,\zeta,\zeta'} \mathrm{d}W_{\zeta'}. \tag{5.15}$$

As an application, consider the problem of characterizing a *positive* square-integrable martingale M_z on the sample space of a Wiener process. From [4] we know that M has a representation of the form

$$M_z = M_0 + \int_{R_z} \phi_\zeta \mathrm{d}W_\zeta + \int_{R_z \times R_z} \psi_{\zeta,\zeta'} \mathrm{d}W_\zeta \mathrm{d}W_{\zeta'} \tag{5.16}$$

without loss of generality we can assume $M_0 = 1$. Now, suppose ϕ and ψ are almost surely bounded. Then, write

$$M_z = 1 + \int_{R_z} u(z,\zeta') \mathrm{d}W_{\zeta'} = 1 + \int_{R_z} \tilde{u}(z,\zeta) \mathrm{d}W_\zeta \tag{5.17}$$

where

$$u(z,\zeta') = \phi_{\zeta'} + \int_{R_z} I(\zeta \curlywedge \zeta') \psi_{\zeta,\zeta'} \mathrm{d}W_\zeta,$$

$$\tilde{u}(z,\zeta) = \phi_\zeta + \int_{R_z} I(\zeta \curlywedge \zeta') \psi_{\zeta,\zeta'} \mathrm{d}W_{\zeta'}. \tag{5.18}$$

Equation (5.12) now yields

$$\ln M_z = \int_{R_z} [u(z,\zeta')/M_{\zeta'\otimes z}] \mathrm{d}W_{\zeta'} - \tfrac{1}{2} \int_{R_z} [u(z,\zeta')/M_{\zeta'\otimes z}]^2 \mathrm{d}\zeta' \tag{5.19}$$

The second equation in (5.17) yields

$$M_{\zeta'\otimes z} = M_{\zeta'} + \int_{R_z} I(\zeta \curlywedge \zeta') \tilde{u}(\zeta'\otimes\zeta, \zeta) \mathrm{d}W_\zeta. \tag{5.20}$$

The first equation in (5.18) can now be used with (5.20) to yield

$$h(z,\zeta') = [u(z,\zeta')/M_{\zeta'\otimes z}] = \alpha_{\zeta'} + \int_{R_z} \beta_{\zeta,\zeta'} [\mathrm{d}W_\zeta - \tilde{h}(\zeta \vee \zeta', \zeta) \mathrm{d}\zeta],$$

where

$$\alpha_{\zeta'} = (\phi_{\zeta'}/M_{\zeta'}), \qquad \tilde{h}(z,\zeta) = \tilde{u}(z,\zeta)/M_{z\otimes\zeta}$$

and

$$\beta_{\zeta,\zeta'} = [(\psi_{\zeta,\zeta'}/M_{\zeta\vee\zeta'}) - h(\zeta \vee \zeta', \zeta') \tilde{h}(\zeta \vee \zeta', \zeta)] I(\zeta \vee \zeta').$$

We now have the following alternative representations for M_z:

$$M_z = \exp\left\{ \int_{R_z} h(z,\zeta') \mathrm{d}W_{\zeta'} - \tfrac{1}{2} \int_{R_z} h^2(z,\zeta') \mathrm{d}\zeta' \right\},$$

$$M_z = \exp\left\{ \int_{R_z} \tilde{h}(z,\zeta) \mathrm{d}W_\zeta - \tfrac{1}{2} \int_{R_z} \tilde{h}^2(z,\zeta) \mathrm{d}\zeta \right\},$$

$$M_z = \exp\left\{ \int_{R_z} \alpha_\zeta \, dW_\zeta + \int_{R_z \times R_z} \beta_{\zeta,\zeta'} \, dW_\zeta \, dW_{\zeta'} \right.$$
$$- \tfrac{1}{2} \int_{R_z} \alpha_\zeta^2 \, d\zeta - \tfrac{1}{2} \int_{R_z \times R_z} \beta_{\zeta,\zeta'}^2 \, d\zeta \, d\zeta'$$
$$- \int_{R_z \times R_z} \beta_{\zeta,\zeta'} [h(\zeta \vee \zeta', \zeta') \, dW_\zeta \, d\zeta' + \tilde{h}(\zeta \vee \zeta', \zeta) \, d\zeta \, dW_{\zeta'}$$
$$\left. - h(\zeta \vee \zeta', \zeta') \tilde{h}(\zeta \vee \zeta', \zeta) \, d\zeta \, d\zeta'] \right\},$$

$$M_z = 1 + \int_{R_z} \alpha_\zeta M_\zeta \, dW_\zeta$$
$$+ \int_{R_z \times R_z} M_{\zeta \vee \zeta'} [\beta_{\zeta,\zeta'} + h(\zeta \vee \zeta', \zeta') \tilde{h}(\zeta \vee \zeta', \zeta)] \, dW_\zeta \, dW_{\zeta'}.$$

The functions $h, \tilde{h}$ are related to α and β by the equations

$$h(z, \zeta') = \alpha_{\zeta'} + \int_{R_z} \beta_{\zeta,\zeta'} [dW_\zeta - \tilde{h}(\zeta \vee \zeta', \zeta) \, d\zeta],$$

$$\tilde{h}(z, \zeta) = \alpha_\zeta + \int_{R_z} \beta_{\zeta,\zeta'} [dW_{\zeta'} - h(\zeta \vee \zeta', \zeta') \, d\zeta'].$$

The application of these results to transformation of probability measures will be considered in a separate paper.

Finally, we note that the differentiation formulas of Section 4 (4.7 and 4.7′) can be viewed as one-parameter formulas on horizontal and vertical paths, and as such can be generalized to arbitrary increasing paths.

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